

# The Fate of the Universe

How one set of equations changed  
an entire field of science

Brian Kay  
PHY 495

# Objective

- Examine the Friedmann equation and its impact on our understanding of the evolution of the universe
- Produce numerical and analytical solutions to the Friedmann equation
- The results will provide us with the geometry, current age, and ultimate fate of the universe

# Outline

- Part 1 - Early 20<sup>th</sup> century cosmology
- Part 2 – Understanding the Friedmann Equation
- Part 3 – Solving the Friedmann equation

# The Friedmann Equation

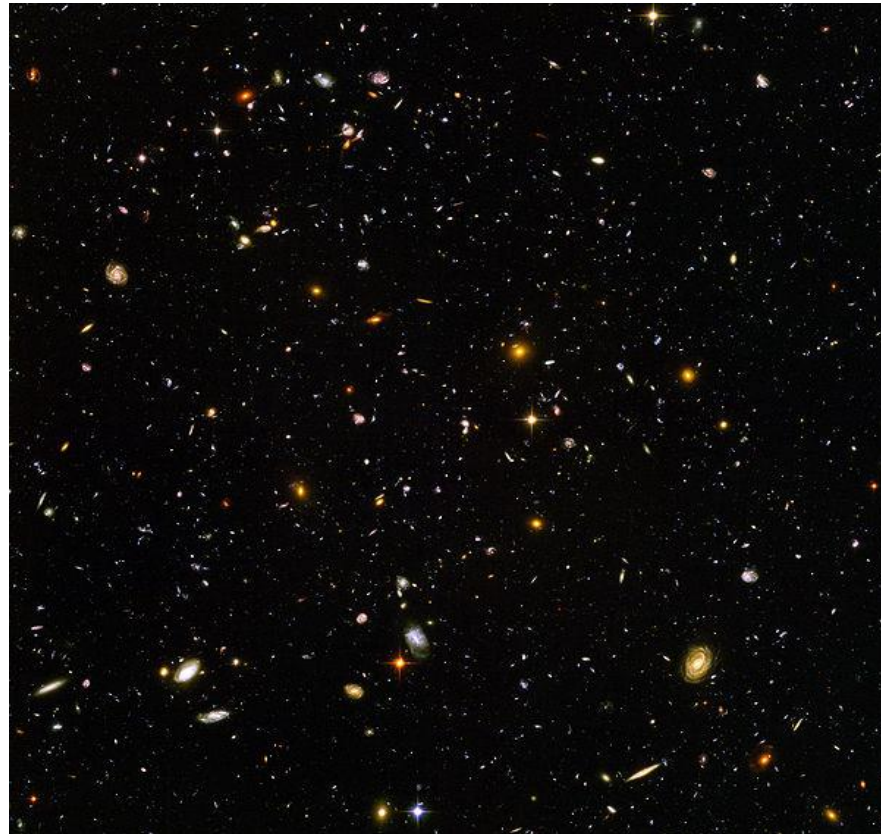
$$H(t)^2 = \frac{8\pi G}{3c^2} \varepsilon - \frac{\kappa c^2}{R_0^2 a(t)^2}$$

$$H_0 t = \int_0^a \frac{da}{[\Omega_{r,0}/a^2 + \Omega_{m,0}/a + \Omega_{\Lambda,0}a^2 + (1-\Omega_0)]^{1/2}}$$

# Part I

## The origins of cosmology

<http://hubblesite.org/newscenter/archive/releases/2004/07>



# Alexander A. Friedmann (1888-1925)

The Man Who Made the Universe Expand

[http://en.wikipedia.org/wiki/File:Aleksandr\\_Fridman.png](http://en.wikipedia.org/wiki/File:Aleksandr_Fridman.png)



*A. Friedmann*

Soviet mathematician and meteorologist

Most famous for contributions to cosmology

First person to mathematically predict an expanding universe (1922)

Derived from Einstein's general relativity

Einstein initially dismissed Friedmann's equations as nothing more than a mathematical "curiosity"

Developed his equation based on an isotropic and homogenous universe

# Expanding Universe

First general relativistic models by Einstein predicted a contracting universe (1917)

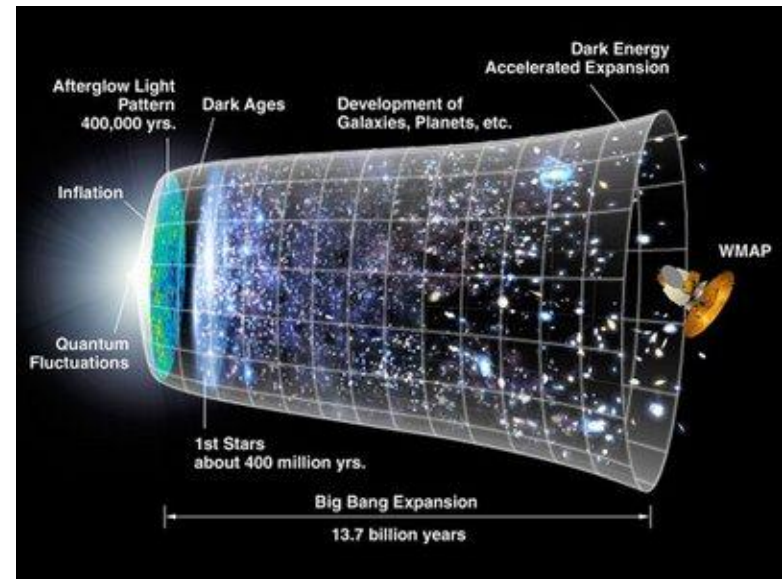
Introduction of Cosmological Constant by Einstein

1922 the Friedmann equation predicted an expanding universe

1928 Georges Lemaitre independently concluded the same

Lemaitre was the first to propose a 'primeval atom' theory, later known as the Big Bang

[http://en.wikipedia.org/wiki/File:CMB\\_Timeline300\\_no\\_WMAP.jpg](http://en.wikipedia.org/wiki/File:CMB_Timeline300_no_WMAP.jpg)



# Edwin Hubble

- Confirmed the existence of other galaxies besides the Milky way (1924)
- Vesto Slipher first discovered the Doppler shift in the spectra of galaxies, Hubble used this to prove the universe is expanding (1929)
- Velocities of receding galaxies from Earth are proportional to their distance from Earth
- Caused Einstein to abandon his cosmological constant

<http://en.wikipedia.org/wiki/File:Hubble.jpg>



$$v = H_0 D$$

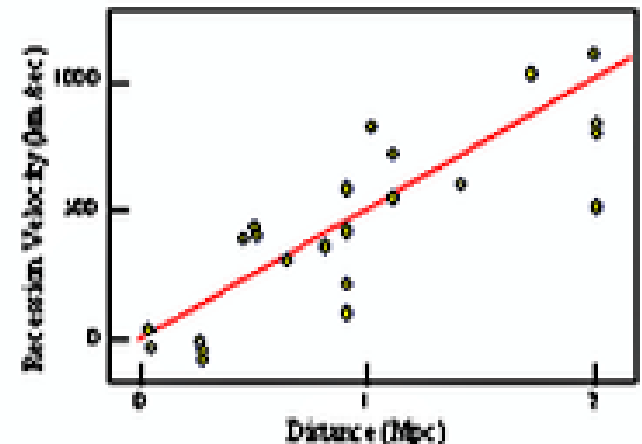


# Hubble's Constant

$$H_0 = 70 \pm 7 \text{ km s}^{-1} \text{ Mpc}^{-1}$$

- Gives the speed of a galaxy in km/s that is 1 Mpc away
- The inverse  $H_0^{-1}$  is known as Hubble time referred to as the expansion time scale
- Hubble first calculated  $H_0$  to be  $500 \text{ km s}^{-1} \text{ Mpc}^{-1}$
- $H_0 = 2.26 \times 10^{-18} \text{ s}^{-1}$
- $H_0^{-1} \approx 14 \text{ Gyr}$

## Hubble's Data (1929)



$$H_0 = \left. \frac{\dot{a}}{a} \right|_{t=t_0}$$

$$H_0 = \frac{v_p(t_0)}{d_p(t_0)} = \frac{d_p'(t_0)}{d_p(t_0)}$$

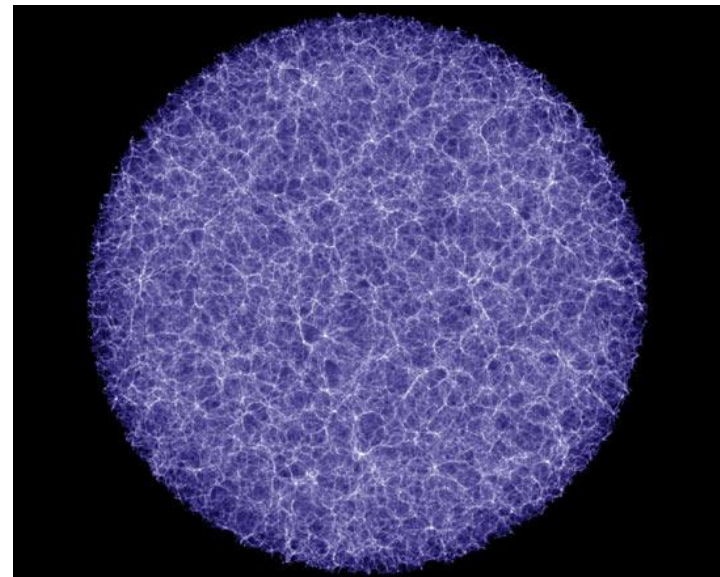
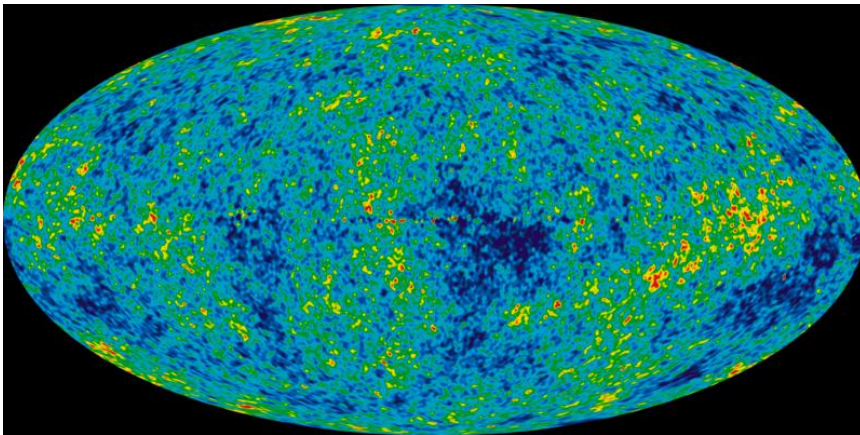
$$d_p' = \frac{\dot{a}}{a} d_p$$

# Cosmic Microwave Background Radiation (CMB)

- Radiation from the “Big Bang” first theorized in 1948 by Gamow, Alpher, and Herman
- Steady State (SS) was an alternative theory developed in 1948 by Hoyle, Gold, and Bondi
- SS claimed that new matter was continuously being created as the universe expanded to keep the average density of matter equal over time
- Discovered in 1964 by radio astronomers Arno Penzias and Robert Wilson
- CMB caused most cosmologists to accept “Big Bang” and reject SS
- COBE (1989-1996), WMAP (2003-2010), Planck Spacecraft -No data yet
- Uniform distribution suggesting an isotropic and homogeneous universe
- The mean temperature is  $2.725 \pm 0002$  K

[http://www.daviddarling.info/encyclopedia/L/large-scale\\_structure.html](http://www.daviddarling.info/encyclopedia/L/large-scale_structure.html)

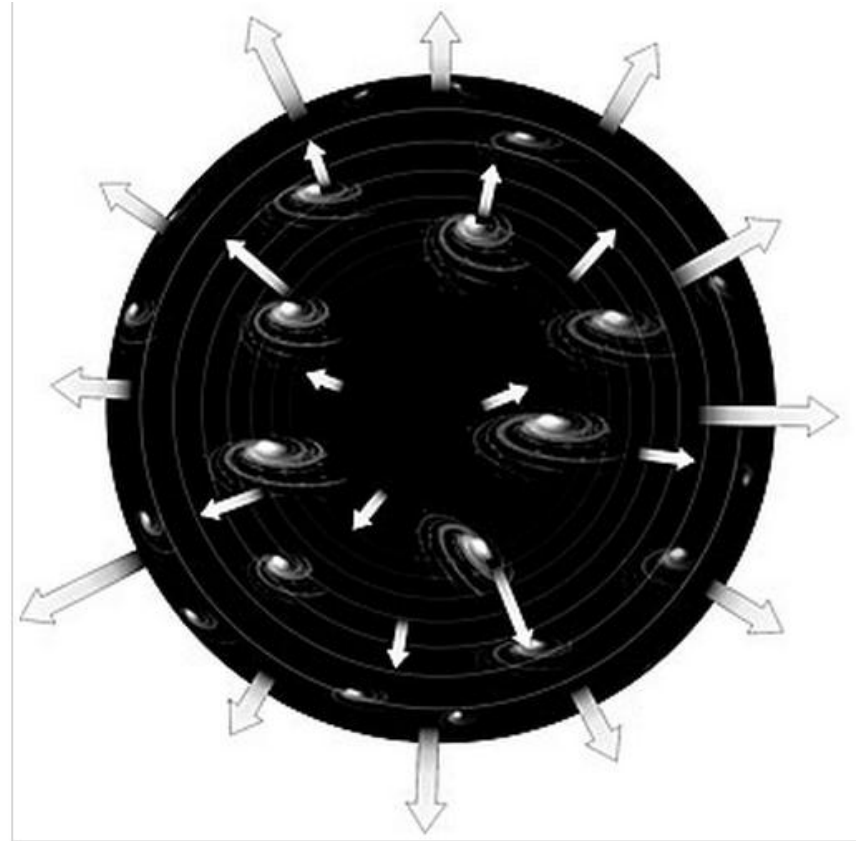
<http://map.gsfc.nasa.gov/media/060913/index.html>



# Rate of Expansion

<http://scienceblogs.com/startswithabang/2010/01>

- Two teams in the 1990's set out to show the rate deceleration of the universe
- They both independently concluded the expansion is accelerating
- Appears to be a repulsive force countering gravity now dubbed dark energy
- Now cosmologists are reevaluating the importance of the cosmological constant



# Part II

## Understanding the Friedmann Equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \varepsilon - \frac{kc^2}{R_0^2 a^2}$$

# Robertson-Walker Metric

$$ds^2 = (cdt)^2 - a^2(t)[d\chi^2 + S_K^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2)]$$

- Derived from Einstein's general relativity
- A space-time metric that satisfies the principle of a homogeneous, isotropic universe
- The geometric structure should remain constant with time in comoving coordinates  $(r, \theta, \phi)$  and physical distances scale with  $a(t)$
- Represents the  $\kappa$  value in the Friedmann equation with values of either  $(1, -1, 0)$

$$K = \begin{cases} 1 = open \\ 0 = flat \\ -1 = closed \end{cases}$$

$$S_K = \begin{cases} R \sin(r/R) & (K = +1) \\ r & (K = 0) \\ R \sinh(r/R) & (K = -1) \end{cases}$$

# Cosmic Scale Factor $a(t)$

$$H(t)^2 = \frac{8\pi G}{3c^2} \epsilon - \frac{\kappa c^2}{R_0^2 a(t)^2}$$

$$H(t) = \frac{\dot{a}}{a}$$

$$H_0 = H(t_0) = \left. \left( \frac{\dot{a}}{a} \right) \right|_{t=t_0}$$

- Represents relative expansion of the universe
- Dimensionless function of time
- Also called the Robertson-Walker scale factor
- Relates proper distance between a pair of objects undergoing isotropic and homogeneous expansion

$$d(t) = a(t)d_0$$

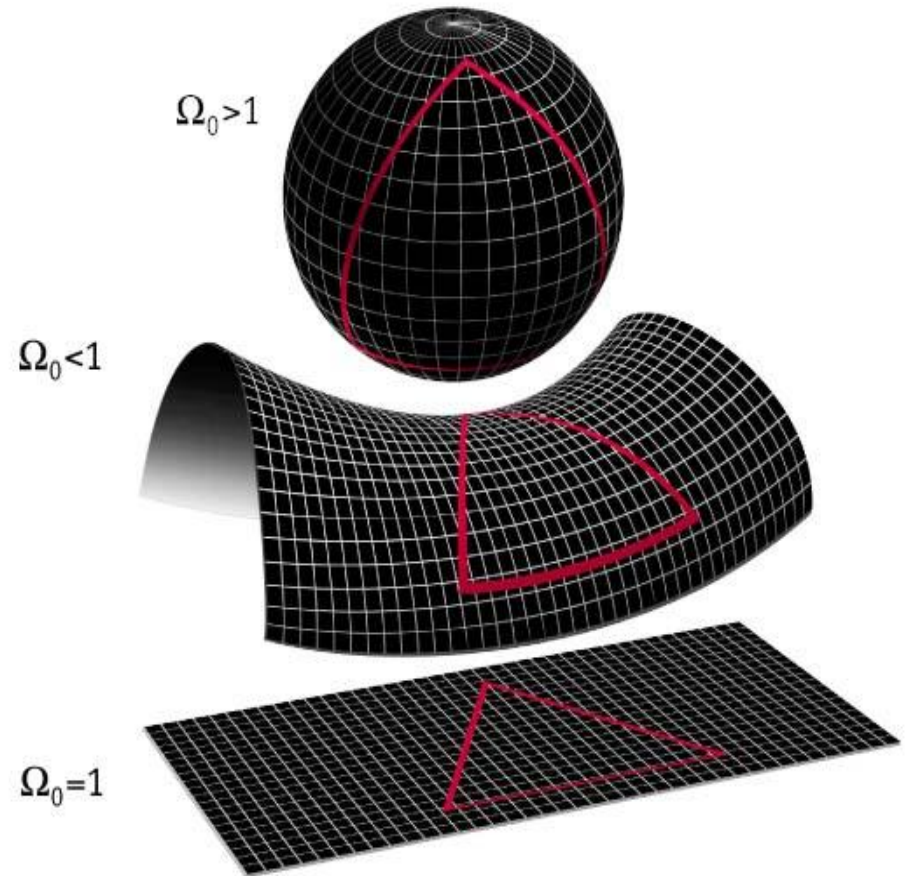
$$a(t_0) = 1$$

# Curvature

$$H(t)^2 = \frac{8\pi G}{3c^2} \epsilon - \frac{\kappa c^2}{R_0^2 a(t)^2}$$

<http://map.gsfc.nasa.gov/media/990006/index.html>

- Open, closed, flat
- Data suggests we live in a flat universe
- Derived from Robertson-Walker metric
- $K$  is the curvature constant
- $R_0$  is the radius of curvature
- Determines how the universe will end



MAP990006

# Density is Destiny!

- Consists of matter, radiation, and most likely a cosmological constant ( $\Lambda$ )
- Einstein's biggest blunder might have been correct all along. (Dark energy)
- More convenient to use the ratio of the density to the critical density
- $\kappa / R_0 = 0$  gives the critical density
- The density parameter  $\Omega$  ultimately governs whether the curvature is:
  - negative ( $\Omega < 1$ )
  - positive ( $\Omega > 1$ )
  - flat ( $\Omega = 1$ )

$$H(t)^2 = \frac{8\pi G}{3c^2} \varepsilon - \frac{\kappa c^2}{R_0^2 a(t)^2}$$

$$\varepsilon_{c,0} = \frac{3c^2 H_0^2}{8\pi G}$$

$$\Omega(t) = \frac{\varepsilon(t)}{\varepsilon_c(t)}$$



# Rewriting the Friedmann equation in integral form

$$H(t)^2 = \frac{8\pi G}{3c^2} \varepsilon - \frac{\kappa c^2}{R_0^2 a(t)^2}$$

$$\varepsilon_m = \frac{\varepsilon_{m,0}}{a^3}, \quad \varepsilon_r = \frac{\varepsilon_{r,0}}{a^4}, \quad \varepsilon_{\Lambda,0} = \text{constant}$$

$$\varepsilon_{c,0} = \frac{3c^2 H_0^2}{8\pi G}, \quad \Omega(t) = \frac{\varepsilon(t)}{\varepsilon_c(t)}$$

$$\Omega_{m,0} = \frac{\varepsilon_{m,0}}{\varepsilon_{c,0}}, \quad \Omega_{r,0} = \frac{\varepsilon_{r,0}}{\varepsilon_{c,0}}, \quad \Omega_{\Lambda,0} = \frac{\varepsilon_{\Lambda,0}}{\varepsilon_{c,0}}$$

$$\frac{\kappa}{R_0^2} = \frac{H_0^2}{c^2} (\Omega_0 - 1)$$

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2}$$

$$H = \frac{\dot{a}}{a}$$

$$H(t)^2 = \frac{8\pi G}{3c^2} \varepsilon(t) - \frac{H_0^2}{a(t)^2} (\Omega_0 - 1)$$

$$H_0^{-1} \dot{a} = \left[ \frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0} a^2 + (1 - \Omega_0) \right]^{1/2}$$

$$\frac{H(t)^2}{H_0^2} = \frac{\varepsilon(t)}{\varepsilon_{c,0}} + \frac{(1-\Omega_0)}{a(t)^2}$$

$$H_0 t = \int_0^a \frac{da}{\left[ \frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0} a^2 + (1-\Omega_0) \right]^{1/2}}$$

# Part III

Analytic and numerical solutions to  
the Friedmann equation

# Matter Only ( $\kappa = 0$ )

## Einstein-de Sitter

$$\Omega_0 = \Omega_m + \Omega_r + \Omega_\Lambda = 1$$

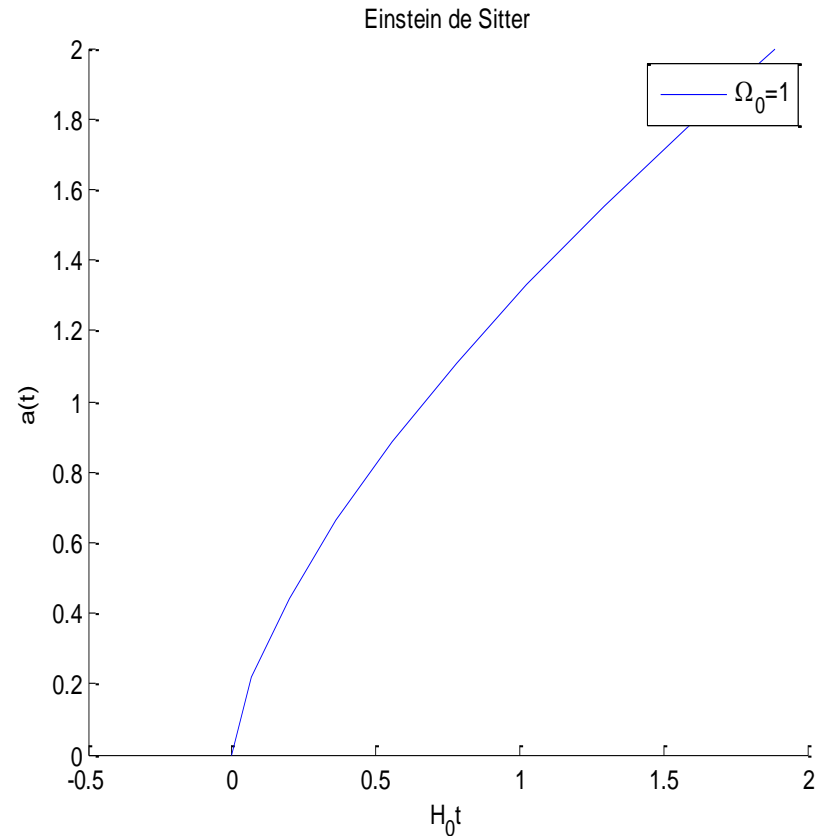
$$\Omega_0 = \Omega_{m,0} = 1 \quad \Omega_{r,0} = \Omega_{\Lambda,0} = 0$$

$$H_0 t = \int_0^a \frac{da}{[\Omega_{r,0}/a^2 + \Omega_{m,0}/a + \Omega_{\Lambda,0}a^2 + (1-\Omega_0)]^{1/2}}$$

$$H_0 t_0 = \int_0^1 \frac{da}{[1/a]^{1/2}} = \frac{2}{3}$$

$$t_0 = \frac{2}{3H_0}$$

$$t_0 = \frac{2}{3 \cdot 0.07256} = 9.187 \text{ Gyr}$$



$$a(\theta) = \frac{1}{2} \frac{\Omega_0}{\Omega_0 - 1} (1 - \cos \theta), \quad \text{Matter Only } (\kappa = 1)$$

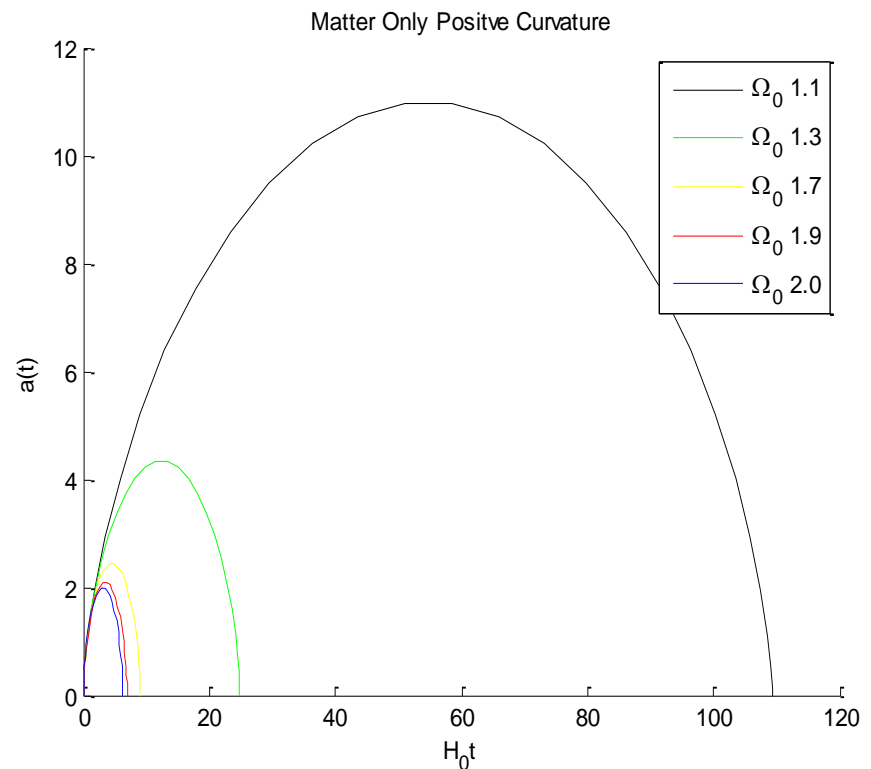
$$t(\theta) = \frac{1}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} (\theta - \sin \theta) \quad \text{Exact Solution}$$

- These are closed universes with a finite time
- At some time in the future expansion will cease and all matter will collapse in on itself called the “Big Crunch”

$$(\Omega_0 > 1, \kappa = 1)$$

$$H_0 t_0 = \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} \cos^{-1} \left( \frac{2 - \Omega_0}{\Omega_0} \right) - \frac{1}{\Omega_0 - 1}$$

$\Omega_0$	$t_0$ (Gyr)	$t_{\text{collapse}}$ (Gyr)
1.1	9.011	1505.011
1.3	8.697	342.355
1.7	8.182	125.607
1.9	7.966	96.295
2.0	7.867	86.589



# Matter Only ( $\kappa = -1$ )

## Exact Solution

$$(\Omega_0 < 1, \kappa = -1)$$

$$a(\eta) = \frac{1}{2} \frac{\Omega_0}{(1-\Omega_0)} (\cosh \eta - 1)$$

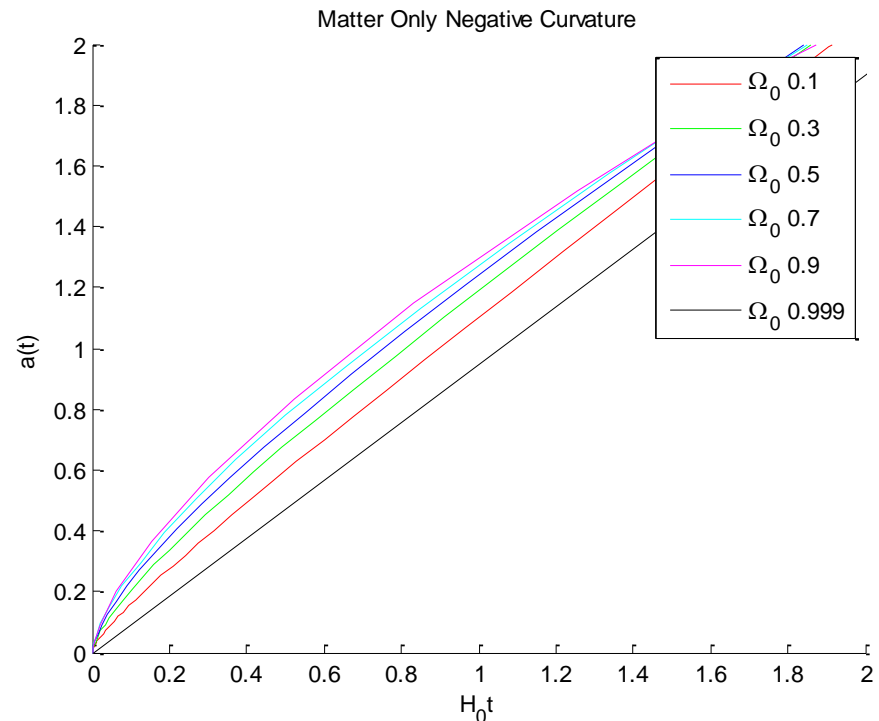
$$t(\eta) = \frac{1}{2} \frac{\Omega_0}{(1-\Omega_0)^{3/2}} (\sinh \eta - \eta)$$

- Expansion will continue for all time. Universe ends in the “Big Chill”

$$H_0 t_0 = \frac{1}{1-\Omega_0} - \frac{\Omega_0}{2(1-\Omega_0)^{3/2}} \cosh^{-1} \left( \frac{2-\Omega_0}{\Omega_0} \right)$$

$$\Omega_0 = 0.1, 0.3, 0.5, 0.7, 0.9, 0.999$$

$\Omega_0$	$t_0$ (Gyr)
0.1	12.377
0.3	11.146
0.5	10.384
0.7	9.824
0.9	9.379
0.999	9.189



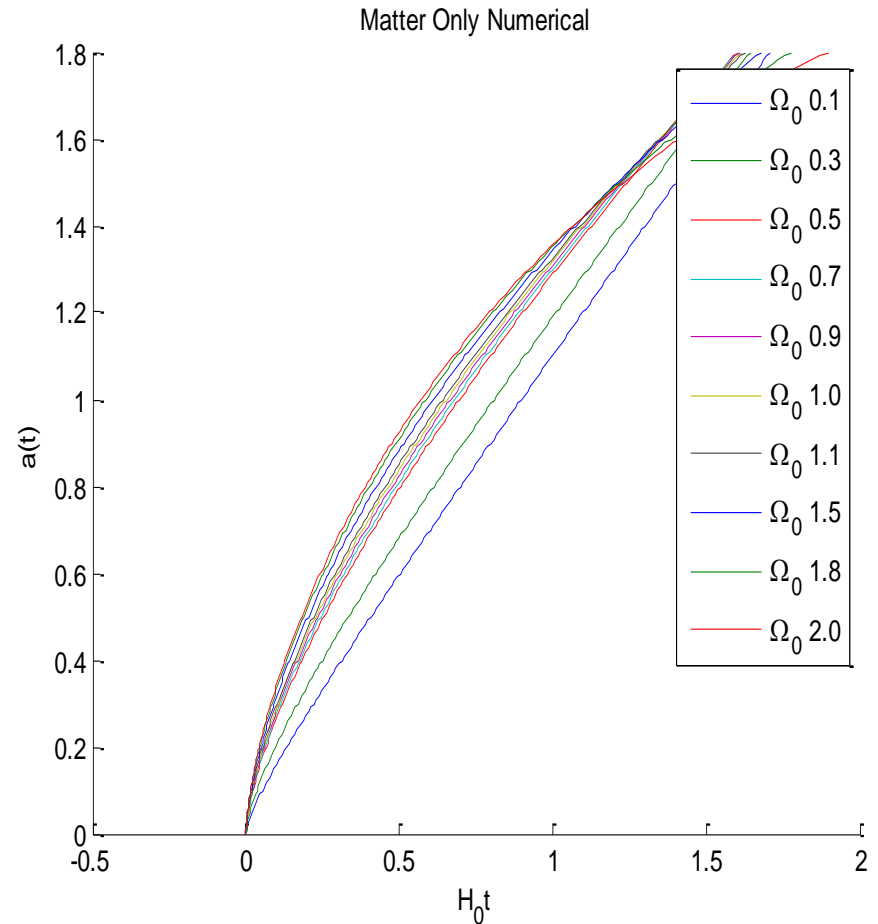
# Matter Only

## Numerical Solutions

$$H_0 t = \int_0^a \frac{da}{[\Omega_{r,0}/a^2 + \Omega_{m,0}/a + \Omega_{\Lambda,0}a^2 + (1-\Omega_0)]^{1/2}}$$

$$\Omega_0 = \Omega_{m,0} \quad H_0 t = \int_0^1 \frac{da}{[\frac{\Omega_{m,0}}{a} + (1-\Omega_0)]^{1/2}}$$

$\Omega_0$	$t_0$ (Gyr)	
	Analytical	Numerical
0.1	12.377	12.377
0.3	11.146	11.146
0.5	10.384	10.385
0.7	9.824	9.824
0.9	9.379	9.379
1.0	9.187	9.187
1.1	9.011	9.013
1.3	8.697	8.697
1.7	8.182	8.183
1.9	7.966	7.967
2.0	7.867	7.867



# Matter + $\Lambda$ ( $\kappa = 0$ )

$$\Omega_0 = \Omega_m + \Omega_r + \Omega_\Lambda = 1$$

$$\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$$

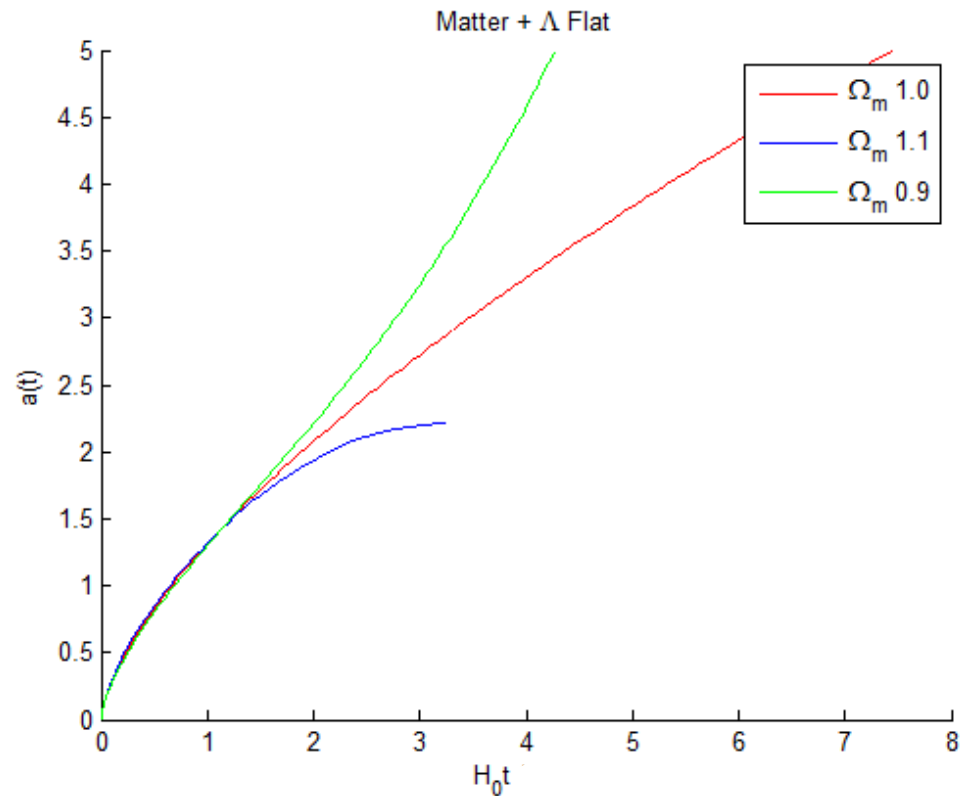
$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + (1 - \Omega_{m,0})$$

If  $\Omega_{m,0} > 1$  then  $\Omega_{\Lambda,0} < 0$

If  $\Omega_{m,0} < 1$  then  $\Omega_{\Lambda,0} > 0$

$\Omega_{m,0}$	$\Omega_{\Lambda,0}$	$\Omega_0$
0.9	0.1	1
1	0	1
1.1	-0.1	1

$$H_0 t = \int_0^a \frac{da}{[\Omega_{r,0}/a^2 + \Omega_{m,0}/a + \Omega_{\Lambda,0}a^2 + (1 - \Omega_0)]^{1/2}}$$



# Matter + $\Lambda$ ( $\kappa = 0$ )

$$\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$$

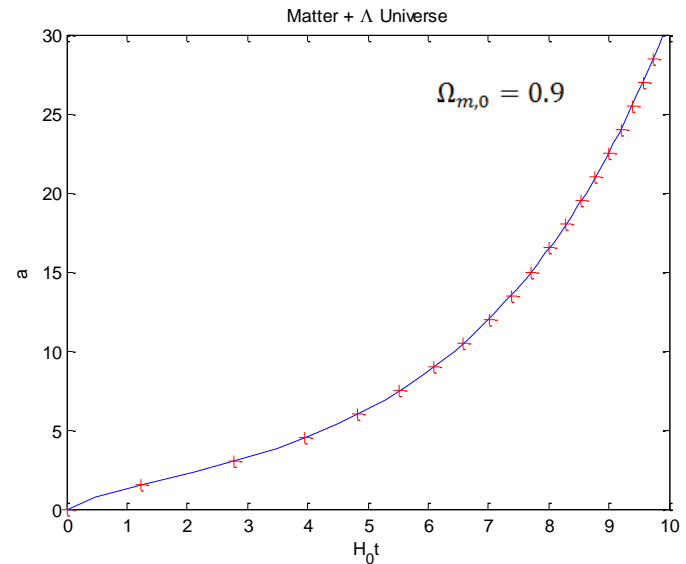
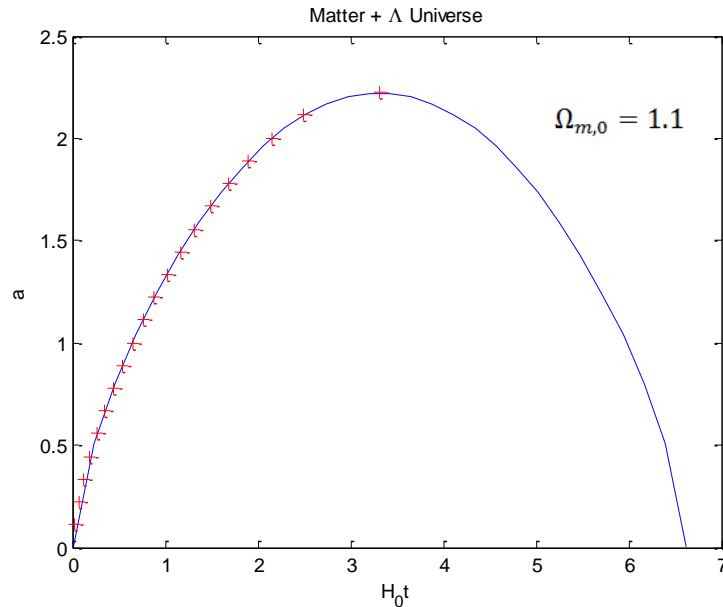
$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + (1 - \Omega_{m,0})$$

$$\Omega_{\Lambda,0} < 0 \quad a_{max} = \left( \frac{\Omega_{m,0}}{\Omega_{m,0} - 1} \right)^{1/3}$$

$$\Omega_{\Lambda,0} > 0$$

$$t_{crunch} = \frac{2\pi}{3H_0\sqrt{\Omega_{m,0}-1}} \quad H_0 t = \frac{2}{3\sqrt{\Omega_{m,0}-1}} \sin^{-1} \left[ \left( \frac{a}{a_{max}} \right)^{3/2} \right]$$

$$t_0 = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}} \ln \left[ \frac{\sqrt{1-\Omega_{m,0}} + 1}{\sqrt{\Omega_{m,0}}} \right]$$





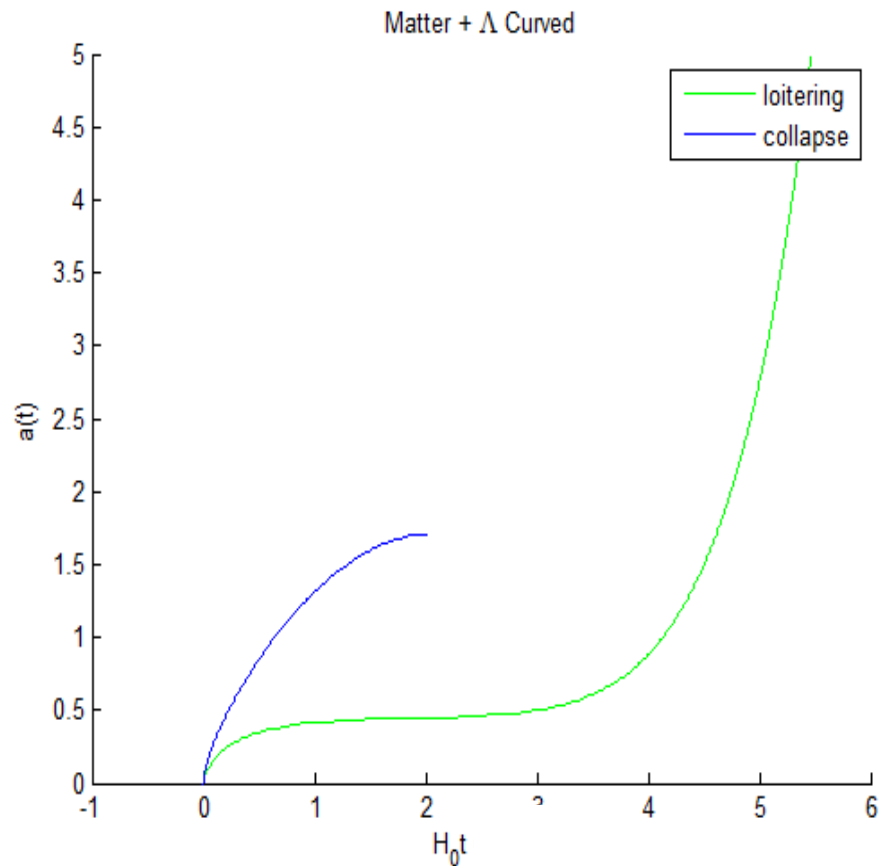
# Matter + $\Lambda$ Curved

$\Omega_{m,0}$	$\Omega_{\Lambda,0}$	$\Omega_0$
0.3	1.713	2.013
1	-0.3	0.7

$$H_0 t = \int_0^a \frac{da}{\left[ \frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0} a^2 + (1 - \Omega_0) \right]^{1/2}}$$

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1 - \Omega_{m,0} - \Omega_{\Lambda,0}}{a^2}$$

- Loitering is when during expansion  $a$  remains almost constant for a long period of time
- Could be a “big bounce” universe, however, we cannot calculate a present time with no beginning time
- Could lead to unphysical, forbidden range of the scale factor

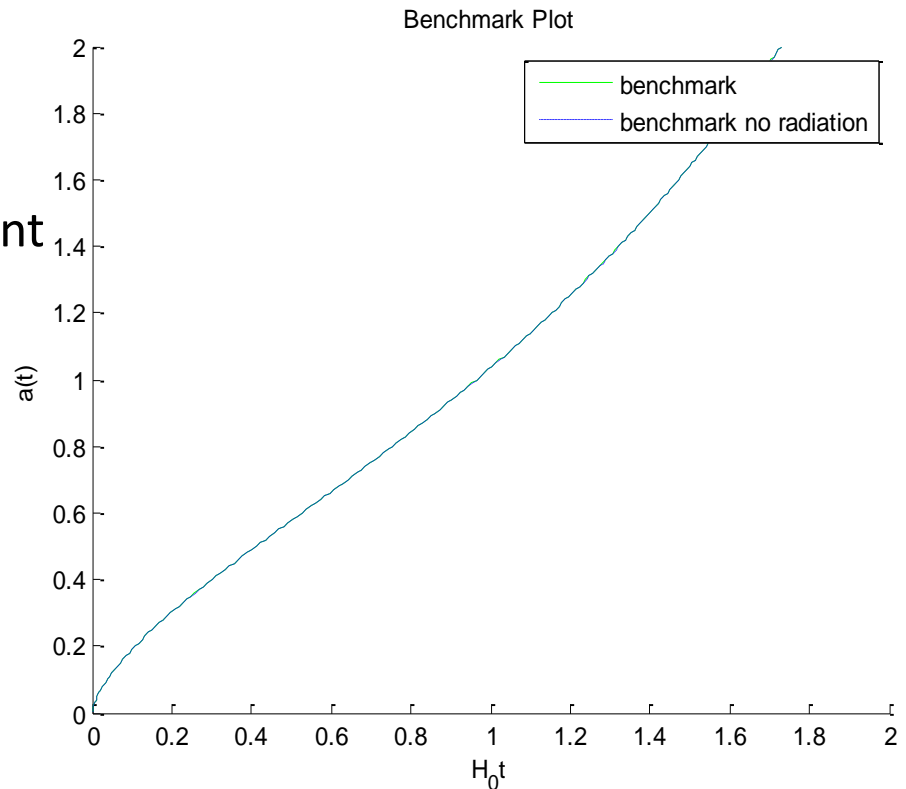


# Benchmark model

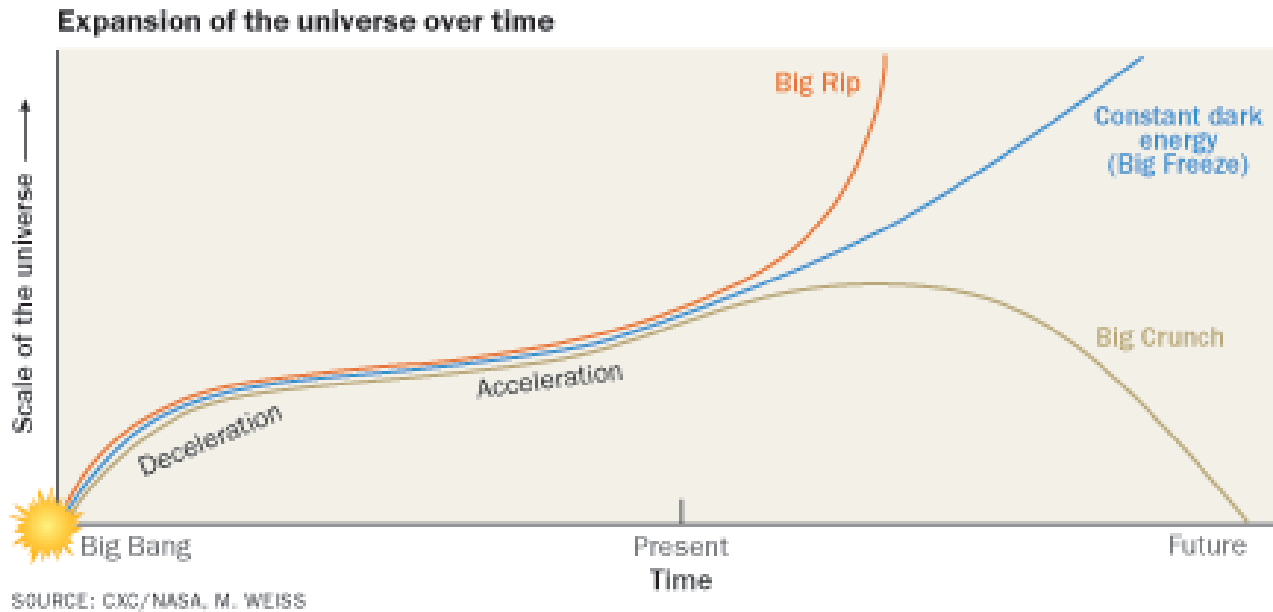
- Considered the best fit to the currently available observational data with  $t_0 = 14\text{Gyr}$
- Is spatially flat and contains radiation, matter, and a dominant cosmological constant

$\Omega_{m,0}$	$\Omega_{\Lambda,0}$	$\Omega_{r,0}$	$\Omega_0$
0.3	0.7	0	1
0.3	0.7	$8.4\text{e-}5$	1.000084

$$H_0 t = \int_0^a \frac{da}{[\Omega_{r,0}/a^2 + \Omega_{m,0}/a + \Omega_{\Lambda,0}a^2 + (1 - \Omega_0)]^{1/2}}$$



# Conclusion



[www.sciencenews.org/view/download/id/72459/name/Cosmic\\_Armageddon\\_](http://www.sciencenews.org/view/download/id/72459/name/Cosmic_Armageddon_)

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- Tropp, Eduard A. *Alexander a Friedmann The Man Who Made the Universe Expand*. Cambridge: Cambridge Univ Pr, 2006. Print.