RELATIVITY AND THE SEARCH FOR GRAVITATIONAL WAVES
Outline

I. History of Motion until Special Relativity
II. Special and General Theories and Their Implications
III. LIGO, LISA and Gravitational Wave Detection.
Galileo and the Principle of Relativity

Galilean Relativity

- Any two observers moving at constant speed and direction with respect to one another will obtain the same results for all mechanical experiments.

Galileo Galilei (1564 – 1642)
Parable of the Ship

“Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals…

...With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. ...

...have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that.

You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still.”
Man Inside a Train Car

Throws a ball at speed $v$, observes it to move with that same speed $v$. 
Outside the Train Car

An observer outside sees the ball moving with a speed equal to the speed of the ball relative to the train \( v \), plus the speed of the train relative to the ground \( U \).

\[ \text{speed} = v + U \]
Galilean Transformation

\[ x' = x - Ut \]
\[ y' = y \]
\[ z' = z \]
\[ t' = t \]
Experiment in two Frames of Reference

To a “Stationary” Observer

To a man inside the car, a ball thrown upward has only a vertical displacement.

\[
\frac{d^2 x'}{dt^2} = a_x' = 0
\]

\[
\frac{d^2 y'}{dt^2} = a_y' = g
\]
Experiment in two Frames of Reference

To a “Moving” Observer

\[
\begin{align*}
\frac{d^2 x}{dt^2} &= \frac{dV_x}{dt} = a_x = 0 \\
\frac{d^2 y}{dt^2} &= a_y = g
\end{align*}
\]

- To a man outside the car, a ball thrown upward has both a vertical and horizontal displacement, but accelerations are observed to be the same.
Newton is considered the father of classical physics and made contributions in several branches of Science, including the development of calculus* and the mathematical description of gravity, force and motion.

*Also attributed to Gottfried Leibniz, 1646-1716.
Newton’s First Law

Stated mathematically as:
\[
\frac{d^2 x}{dt^2} = 0 \quad \frac{d^2 y}{dt^2} = 0 \quad \frac{d^2 z}{dt^2} = 0
\]

The path $S$ through space drawn by a free particle is the shortest distance between two points, a straight line:

\[
S^2 = x^2 + y^2 + z^2
\]

**Line Element:**

\[
dS^2 = dx^2 + dy^2 + dz^2
\]
Newton’s Theory of Gravitation

By equating gravitational and inertial mass,

\[ \vec{F}_{g_{2 \rightarrow 1}} \equiv \vec{F}_{2} \propto \frac{m_{2}m_{1}}{r^{2}} \hat{r} = \frac{Gm_{2}m_{1}}{r^{2}} \hat{r} = m_{2} \vec{a}_{2} \]

\[ \vec{a}_{2} = \frac{\vec{F}_{2}}{m_{2}} = \frac{Gm_{1}}{r^{2}} \hat{r} \]

\[ G = 6.67 \times 10^{-11} \frac{m^{3}}{kg \cdot s^{2}} \]
So Far:

- The laws of physics are the same as observed in any inertial (non-accelerating) reference frame.
- We can use simple Galilean transformations to move from one inertial coordinate system into another.
- Newton developed a set of laws of motion and gravitation that match experimental observations with near perfect accuracy.
Problem:

- Galilean relativity seemed sufficient until the study of electrodynamics in the 1800’s.
- Systems in electrodynamics were not invariant under Galilean transformations.
Maxwell’s Equations in Empty Space

This set of equations describes the behavior of E&M systems in regions with no electric charge or current

\( (i) \ \nabla \cdot \vec{E} = 0 \)

\( (ii) \ \nabla \cdot \vec{B} = 0 \)

\( (iii) \ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \)

\( (iv) \ \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \)
Decoupling E and B in Maxwell’s Equations

In this form, the equations satisfy a common formula known as the wave equation, describing the properties of a wave propagating through a medium.

\[ \nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \text{and} \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \]

\[ \mu_0 = 4\pi \times 10^{-7} \frac{N}{A^2} \]

\[ \epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{Nm^2} \]

**The Wave Equation**

\[ \nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} \quad \text{where} \; v = \text{wave velocity} \]
“v” in the Wave Equation

\[ \frac{1}{v^2} = \mu_0 \varepsilon_0 \rightarrow v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \]

\[ \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{(4\pi \times 10^{-7})(8.85 \times 10^{-12})}} \approx 3 \times 10^8 \text{ m/s} \]

The speed of light! This is how Maxwell first inferred that light is an electromagnetic wave. But through what medium is it propagating?
Light travels along equal path lengths in perpendicular directions. The difference in motion relative to the ether should cause the two waves to travel at slightly different speeds, and as such be out of phase with each other and interfere.

Albert Michelson, left, 1852-1931
And Edward Morley, 1838-1923
For the given apparatus with an effective arm length of 11m, Michelson and Morley expected to see a shift of about 0.4 fringes. What they observed was < 0.01.
The Most Famous Failed Experiment

Attempts were made to explain the findings. Most notable was that of **Fitzgerald-Lorentz Contraction**:

That objects will physically contract along the direction of motion by the factor gamma, in agreement with the new Lorentz transformations of electrodynamics:

\[
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\]
Systems in electrodynamics were *not* invariant under Galilean transformations, but they *were* invariant under Lorentz transformations. A complete explanation for the reason these transformations work would be left for Einstein to give.

\[
\begin{align*}
x' &= \gamma(x - vt) \\
y' &= y \\
z' &= z \\
t' &= \gamma \left( t - \frac{vx}{c^2} \right)
\end{align*}
\]

\[
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

Hendrik Antoon Lorentz
1853-1928
Einstein dismissed the notion of an ether altogether, and instead accepted that the speed of light is invariant under all coordinate transformations.
Now the train observer measures a speed $c$.

The ground observer will *not* measure a speed of $c + U$.

He also measures a speed $c$, as does an observer on a train travelling in the opposite direction!
Back to the train

\[ \Delta t' = \frac{d}{c} \implies d = c\Delta t' \]
Back to the train

\[(c\Delta t)^2 = (v\Delta t)^2 + d^2\]
Time Dilation

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

*note:* \( \gamma \geq 1 \)

for \( v \ll c, \quad \gamma \to 1 \)

for \( v \to c, \quad \gamma \to \infty \)

\[ (c\Delta t)^2 = (v\Delta t)^2 + (c\Delta t')^2 \]

\[ (c\Delta t)^2 - (v\Delta t)^2 = (c\Delta t')^2 \]

\[ \sqrt{c^2 \Delta t^2 - v^2 \Delta t^2} = c\Delta t' \]

\[ c\Delta t \sqrt{1 - \frac{v^2}{c^2}} = c\Delta t' \]

\[ \Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ \Delta t = \gamma \Delta t' \]
A New Geometry is Needed

- Einstein’s revelation was that we must dispense with the idea of an absolute fixed time. There is no universal clock, and we move not through 3-D space but through 4-D spacetime.
- At velocities comparable to that of light, this fact becomes more noticeable.
- What are the implications of accepting this notion?
A New Geometry is Needed

We need a line element in which the distance between two events in spacetime is invariant, similar to the line element of 3D Newtonian space.

\[(x, y, z) \leftrightarrow (x', y', z')\] \[(t, x, y, z) \leftrightarrow (t', x', y', z')\]

\[dS^2 = dx^2 + dy^2 + dz^2\] \[ds^2 = ?\]

We can use the Lorentz transformations to find the distance between two events from two different frames.
A New Geometry is Needed

For convenience we can choose the origins of our frames to coincide without any loss of generality.

\[ \Delta t = t - t_0 \quad \text{let } t_0 = 0 \implies \Delta t = t, \quad \Delta x = x, \text{etc.} \]

\[ \Delta t' = t' - t'_0 \quad \text{let } t'_0 = 0 \implies \Delta t' = t', \Delta x' = x', \text{etc.} \]

And noting that:

\[ y' = y \]

\[ z' = z \]

We will temporarily ignore these two sets of coordinates.
A New Geometry is Needed

\[-(ct')^2 + x'^2 = -c^2 \gamma^2 \left( t - \frac{v}{c^2} x \right)^2 + \gamma^2 (x - vt)^2 \]

\[= \gamma^2 \left[ -c^2 \left( t^2 - \frac{2vtx}{c^2} + \frac{v^2x^2}{c^4} \right) + x^2 - 2vtx + v^2 t^2 \right] \]

\[= \gamma^2 \left[ -(ct)^2 + (vt)^2 + x^2 - \left( \frac{vx}{c} \right)^2 \right] \]

\[= \gamma^2 \left( 1 - \frac{v^2}{c^2} \right) [-(ct)^2 + x^2] \]

\[= -(ct)^2 + x^2 \]

\[-(ct')^2 + x'^2 = -(ct)^2 + x^2 \]

\[-(ct')^2 + x'^2 + y'^2 + z'^2 = S^2 = -(ct)^2 + x^2 + y^2 + z^2 \]
A New Geometry is Needed

We have a new Spacetime Line Element

\[ ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \]

Where \( ds^2 \) is the infinitesimal unit of length. We can also measure infinitesimal units of time by the relation:

\[ d\tau^2 = \frac{-ds^2}{c^2} = -ds^2 \text{ for } c=1. \]
Minkowski Diagram
Minkowski Diagram
Minkowski Diagram
\[ \Delta s^2 = \Delta x^2 = -c^2 \Delta t'^2 + \Delta x'^2 \]
\[ \Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 = 0 \]
Confirmation Of Special Relativistic Predictions

- Muon Experiment:
  At CERN, muons moving at extremely high speeds were measured to have lifetimes about 30 times longer than those that were not.

- Cesium Beam Atomic Clock experiment:
  Clocks were flown aboard commercial airline flights and found to exhibit gravitational and kinematic time dilation in excellent agreement with prediction.
“glückischte Gedanke meines Lebens” -
the happiest thought of my life
The effect of matter/energy is to produce curvature within the local spacetime fabric. The relationship between spacetime curvature and matter energy density is given by the Einstein Equation:

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu} \]
The Einstein Equation

\[ G_{\mu \nu} = 8\pi G T_{\mu \nu} \]

- Represents a system of ten nonlinear partial differential equations
- Solutions for a given distribution of matter/energy yield a description of spacetime curvature in the form of a metric.
Geodesics

The coefficients of our components in flat space can be written in matrix form:

\[
\eta_{\alpha\beta} = \begin{pmatrix}
t & x & y & z \\
t & -1 & 0 & 0 \\
x & 0 & 1 & 0 \\
y & 0 & 0 & 1 \\
z & 0 & 0 & 0 \\
\end{pmatrix}
\]

The *metric* of flat spacetime

So that our line element in flat spacetime can be written:

\[
ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = -dt^2 + dx^2 + dy^2 + dz^2
\]
Geodesics

Or we can look at the time separation:

\[ d\tau^2 = -g_{\alpha\beta} dx^\alpha dx^\beta \]

\[ d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 \]

\[ \eta_{\alpha\beta} = \begin{pmatrix} t & x & y & z \\ t & -1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \\ y & 0 & 0 & 1 & 0 \\ z & 0 & 0 & 0 & 1 \end{pmatrix} \]

The *metric* of flat spacetime

How can we integrate along these infinitesimal components to determine the spacetime curvature?
We define a geodesic to be the path of a free particle between two events in curved spacetime which extremizes the proper time between them.
The proper time between two points can be found through integration:

$$\tau_{AB} = \int_A^B \sqrt{d\tau^2} = \int_A^B \left[ dt^2 - dx^2 - dy^2 - dz^2 \right]^{1/2}$$

We can parameterize the equation:

$$\tau_{AB} = \int_0^1 d\sigma \left[ \left( \frac{dt}{d\sigma} \right)^2 - \left( \frac{dx}{d\sigma} \right)^2 - \left( \frac{dy}{d\sigma} \right)^2 - \left( \frac{dz}{d\sigma} \right)^2 \right]^{1/2}$$

And find the extremal path by taking the Lagrangian:

$$L = \left[ \left( \frac{dt}{d\sigma} \right)^2 - \left( \frac{dx}{d\sigma} \right)^2 - \left( \frac{dy}{d\sigma} \right)^2 - \left( \frac{dz}{d\sigma} \right)^2 \right]^{1/2} \to \frac{\partial L}{\partial x^\alpha} - \frac{\partial}{\partial \sigma} \left( \frac{\partial L}{\partial (dx^\alpha/d\sigma)} \right) = 0$$
The Lagrangian

Note:

\[
\frac{d\tau}{d\sigma} = \frac{d\sigma}{\sqrt{\left(\left(\frac{dt}{d\sigma}\right)^2 - \left(\frac{dx}{d\sigma}\right)^2 - \left(\frac{dy}{d\sigma}\right)^2 - \left(\frac{dz}{d\sigma}\right)^2\right)}} \quad \text{with a little math we can trade parameters}
\]

\[
\frac{d\tau}{d\sigma} = L
\]

For \(x^1 = x\)

\[
\frac{\partial L}{\partial x} = 0 = \frac{d}{d\sigma} \left[ \frac{\partial L}{\partial (dx/d\sigma)} \right] = -\frac{d}{d\sigma} \left[ \frac{1}{2L} \frac{2dx}{d\sigma} \right] = -\frac{d}{d\sigma} \left[ \frac{d\sigma dx}{d\tau d\sigma} \right] = \frac{d^2 x}{d\tau^2}
\]
In flat spacetime, we find a relationship analogous to Newtonian motion:

\[
\frac{d^2 x^\alpha}{d\tau^2} = 0 \quad \text{for} \quad (x^0, x^1, x^2, x^3) = (t, x, y, z)
\]

Using Lagrangian's equations for the line element of a given metric will give us the equations of motion for a particular curved spacetime.

In General:

\[
\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau}
\]
Schwarzschild Geometry:

Solution of Einstein’s equation for spherically symmetric masses

\[
\begin{pmatrix}
t & 0 & 0 & 0 \\
r & 0 & \left[1 - \frac{2M}{r}\right]^{-1} & 0 \\
\theta & 0 & 0 & r^2 \\
\phi & 0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}
\]

\[
ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]

Other metrics describe other spacetime curvatures
Schwarzschild Geometry:

\[
\begin{pmatrix}
  t & -\left[1 - \frac{2M}{r}\right] & 0 & 0 \\
  r & 0 & \left[1 - \frac{2M}{r}\right]^{-1} & 0 \\
  \theta & 0 & 0 & r^2 \\
  \phi & 0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}
\]

Two Christoffel Symbols and Their Equations of Motion:

\[
\Gamma^r_{\theta\theta} = -(r - 2M) \\
\Gamma^r_{\phi\phi} = -\cot \theta \\
\frac{d^2r}{d\tau^2} = (r - 2M) \frac{d\theta}{d\tau} \frac{d\theta}{d\tau} = (r - 2M) \left(\frac{d\theta}{d\tau}\right)^2 \\
\frac{d^2\phi}{d\tau^2} = -2\cot \theta \frac{d\theta}{d\tau} \frac{d\phi}{d\tau}
\]

Other metrics describe other spacetime curvatures
Gravitational Redshift

The frequency of light from a massive object is much higher on its surface than it is when measured far from the source.
Light Deflection

Light passing near a very massive object travels a “straight line” through curved spacetime. Its path will be measurably altered by the local curvature, changing the apparent location of the source.
Gravitational Lensing

Massive objects cause multiple images of the same object by focusing the light from several directions.
Gravitational Lensing
Black Holes

Star that have collapsed upon themselves to create a curvature that not even light will escape.
Gravitational Waves

Masses in accelerated motion produce ripples in the spacetime curvature that propagate away from the source at the speed of light.
What Are Some Sources of Gravitational Waves?

- Collapsing stars
- Supernovae
- Rapidly rotating asymmetric masses
- Binary Star Systems
What Are Some Sources of Gravitational Waves?

Collapsing Star
What Are Some Sources of Gravitational Waves?

Supernova
What Are Some Sources of Gravitational Waves?

Credit: T. Strohmayer and D. Berry
How Could We Detect A Gravitational Wave?

- The orbital period of pulsar PSR B1913+16 in orbit in a binary star system loses 75µs per year from its orbital period-

  This is consistent with the amount of energy it is expected to lose through gravitational radiation.
How Could We Detect A Gravitational Wave?

- Modeling a wave at the source of curvature is extremely complicated.

- Modeling a weak residual ripple based on a linear approximation to a solution of Einstein’s equation is more accessible.
How Could We Detect A Gravitational Wave?

- “Test masses” serve to trace out spacetime curvature in the way a test charge helps detect an electric field: they are small enough as not to produce an appreciable field of their own.

The response of a ring of test masses to a gravitational wave passing perpendicular to the plane of the screen is a linear combination of two polarizations.
How Could We Detect A Gravitational Wave?

Consider the metric: \[ g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \]

Where the first term is our diagonal flat space metric \((-1,1,1,1)\) and the second is a set of small perturbations to that metric, representing a small plane wave propagating in the \(z\) direction.

\[
\eta_{\alpha\beta} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad
h_{\alpha\beta} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix} \alpha \sin(\omega t) \quad |\alpha| \ll 1
\]

Line Element:

\[ ds^2 = -dt^2 + [1 + \alpha \sin(\omega t)]dx^2 + [1 - \alpha \sin(\omega t)]dy^2 + dz^2 \]
How Could We Detect A Gravitational Wave?

- If we could measure tiny changes in distance between test masses, we could detect gravitational waves.

- But what sort of ruler will not simply bend with the waves themselves? And what observer in the ruler’s frame will notice the change?
How Could We Detect A Gravitational Wave?

- Even the wave fronts of light waves have a spatial location at a given instant.

In fact, light waves *are* distorted in gravitational radiation.
How Could We Detect A Gravitational Wave?

Spatial coordinates don’t change, even if the distance does. Take two test masses in a freely falling frame:

In the presence of a wave we can study the equations of motion for the spatial coordinates in our new metric, utilizing its linearity to break up the metric into two parts.
We know the geodesic equation in the unperturbed flat space metric:

\[
\frac{d^2 x^i}{d\tau^2} = -\Gamma^i_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0
\]

The additional perturbation is also zero to first order

The \(u\)'s are the *unperturbed* four-velocities.
3D coordinate separation does not change, but actual distance does.

Take two test masses A and B:

For our metric \( h \) with \( dt\ dy\ dz = 0 \), our distance across the x-axis is:

\[
L(t) = \int_0^{L_x} dx \left[ 1 + h_{xx}(t, 0) \right]^{1/2}
\]

Noting that \( h \ll 1 \), we take a binomial expansion to first order:

\[
[1 + h_{xx}(t, 0)]^{1/2} \approx 1 + \frac{1}{2} h_{xx}(t, 0) \rightarrow 1 + \frac{1}{2} a \sin(\omega t)
\]

So,

\[
L(t) = L_x \left[ 1 + \frac{1}{2} a \sin(\omega t) \right]
\]

Or

\[
\delta L(t) = L_x \frac{1}{2} a \sin(\omega t) \rightarrow \frac{\delta L(t)}{L_x} = \frac{1}{2} a \sin(\omega t)
\]

The first generation of gravitational wave detectors is expected to detect amplitudes on the order of \( 1 \times 10^{-21} \).
So the coordinate positions do not change with respect to each other.

- The *distance* between points *does* change even if the 3D spatial coordinates do not change according to each other.
- Light will still travel at speed $c$ through these new distances.

The path that light takes through this warped spacetime will be altered.
Very sophisticated versions of the Michelson interferometer are in various stages of construction all over the world.
LIGO (Laser Interferometer Gravitational Wave Observatory)

LIGO is a system of two interferometers in Hanford, Washington and one in Livingston, Louisiana. Two interferometers have 4 km arms and one has 2 km arms. The longer the arm, the greater the sensitivity.
The LIGO system is said to contain the largest artificial vacuum in the world.
Other Interferometers

- **TAMA**
  - Japan, 0.3 km arms

- **VIRGO**
  - Italy 3.0 km arms

- **GEO**
  - Germany, 0.6 km arms

- **AIGO**
  - Australia, 0.08 km arms

Together, these observatories can rule out each other’s local disturbances as false signals.
Other Interferometers

**Laser Interferometer Space Antenna**

- LISA will trail the earth by 20 degrees in its trip around the sun, with an arm length of 5 million km.

Each of the three components will be in independent orbit around the sun.
Other Interferometers

- LISA will be able to detect much lower frequency waves than its ground based counterparts, from 0.0001-0.1 Hz, as opposed to 50-1000 Hz (LIGO)
So, Why Look For Them?

- Not just another confirmation of relativity
- Learning to read gravitational waves can give us a vivid new way to look where we could not look before:
  The “death spirals” of stars being swallowed by black holes
  Echoes from the big bang- a “gravitational cosmic background”
- Gravitational radiation is not absorbed and scattered like electromagnetic radiation.
Gravitational waves will provide a whole new brand of astronomy for today’s up and coming scientists...
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